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Generalized Spectral Projections on Symmetric Spaces of Noncompact Type: Paley–Wiener Theorems

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$$f(x)=\frac{1}{(2\pi)^r\,|W|}\int_{\mathfrak{a}_+^*} P_\lambda f(x)\,|c(\lambda)|^{-2}\,d\lambda$$

$$P_\lambda f(x)=(f*\Phi_\lambda)(x),$$

where Φ_λ is the spherical function on X. Taken together they represent, the synthesis and decomposition formulas for appropriate functions f on X in terms of joint eigenfunctions of the invariant differential operators on X. The focus of this paper is the characterization of the range of P_λ on $C_c^\infty(X)$ in the case G has real rank one. This result extends and generalizes a result due to Strichartz on odd-dimensional real hyperbolic space and provides a "spectral" reformulation of the Paley–Wiener theorem for Fourier transform on X due to Helgason. As an application we provide a fairly general support result for the spherical mean operator on X. © 1996 Academic Press, Inc.

1. Introduction

Let G be a connected semisimple Lie group with finite center and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be a Cartan decomposition, so $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup K. Let $G=KAN$, respectively, $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{a}+\mathfrak{n}$, be an Iwasawa decomposition. Here $\mathfrak{a}$ is a maximal Abelian subalgebra of $\mathfrak{p}$, $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$, and A and N are the corresponding analytic subgroups. Let $\mathfrak{a}^*$ and $\mathfrak{a}_c^*$ denote the dual of $\mathfrak{a}$ and the complexification, respectively.

We consider the symmetric space of noncompact type $X=G/K$. Functions f on X will be written as $f(x)$ or, where appropriate, will be identified with a unique right K-invariant function on G, also denoted by $f(g)$, where

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$x = gK$. We denote the origin in X by $0 = eK$ (e is the identity). Let $\Phi_\lambda(\cdot)$ be the spherical function on X. Recall that Φ_λ is the unique K-invariant function on X which is an eigenfunction of all the invariant differential operators and is normalized by $\Phi_\lambda(e) = 1$. Consider the operators P_λ, $\lambda \in \mathfrak{a}_c^*$, defined initially for $f \in C_c^\infty(X)$ via the formula

$$P_\lambda f(x) = (f * \Phi_\lambda)(g) = \int_G f(g_1)\, \Phi_\lambda(g_1^{-1} g)\, dg_1, \tag{1.1}$$

where dg_1 is Haar measure on G normalized as in Section 2. Note that $x \ni x \to P_\lambda f(x)$ is an eigenfunction of every invariant differential operator on X (with the same eigenvalue as Φ_λ), hence we call $P_\lambda f$ the generalized spectral projection of f (although P_λ is certainly not a projection operator). Applying well known formulas for the spherical function, (1.1) can be rewritten as

$$P_\lambda f(g) = \int_K \tilde{f}(\lambda, kM)\, e^{(i\lambda + \rho)(H(g^{-1}k))}\, dk, \tag{1.2}$$

where $H(\cdot)$ is defined through the Iwasawa decomposition ($g = k(g) \exp(H(g))\, n(g)$), dk is the normalized Haar measure on K, and $\tilde{f}$ is the (Helgason) Fourier transform of f [11, 12]. Hence, Helgason's Fourier inversion formula on $C_c^\infty(X)$ rewritten for $P_\lambda f$ is

$$f(x) = \frac{1}{(2\pi)^r\, |W|} \int_{\mathfrak{a}_+^*} P_\lambda f(x)\, |c(\lambda)|^{-2}\, d\lambda, \tag{1.3}$$

where r is the rank of G, $|W|$ is the order of the Weyl group, $\mathfrak{a}_+^*$ is the is a positive Weyl chamber, and $d\lambda$ is the Lebesgue measure on $\mathfrak{a}^*$. In this light, formulas (1.1) and (1.3) represent decomposition and synthesis formulas for functions via joint eigenfunctions of the invariant differential operators on X.

The focus of this paper is the quest to characterize $C_c^\infty(X)$ in terms of properties of the generalized spectral projections, a quest realized in Section 4 for the case rank $G = 1$. Aside from intrinsic interest in this problem from the point of view of spectral theory, our interest derives from the following considerations. First, the point of view of reformulating harmonic analysis in terms of generalized spectral projections was initiated in a series of papers by Strichartz [26–29] and the related work of Agmon [1]. In [26, 27], Strichartz proved a spectral Paley–Wiener theorem for odd-dimensional real hyperbolic spaces and odd-dimensional Euclidean spaces. In [2], the author proved a more general variant of Strichartz's result on Euclidean spaces valid in all dimensions. The variant contained a weaker hypothesis needed for applications described below.

A number of authors have developed Paley–Wiener theorems for the group Fourier transform, e.g., [5, 7, 20]; however, our work is certainly closest to that of Helgason [9]. Helgason's [12] Paley–Wiener theorem for the Fourier transform (rank $G=1$) is an immediate consequence of the "spectral" Paley–Wiener theorem. Lying deeper, and at the heart of the proof of the spectral version, is the fact that the two theorems are (essentially) equivalent. Our methodology relies on explicit structural information about spherical functions on X, due primarily to Helgason [13] and Flensted-Jensen and Koornwinder [9]; this information to date is only known in the case rank $G=1$.

Third, many operators of interest are easily expressed in terms of $P_\lambda f$. For example, consider the K-means of a function defined so that for $a \in A$, $x=gK$, and $g \in G$,

$$M^a f(x) = \int_K f(gka)\, dk \tag{1.4}$$

(in the rank-one case M^a is properly called the spherical mean operator). In terms of $P_\lambda f$, (1.4) reads as

$$M^a f(x) = \frac{1}{(2\pi)^r\, |W|} \int_{\mathfrak{a}_+^*} \Phi_\lambda(a)\, P_\lambda f(x)\, |c(\lambda)|^{-2}\, d\lambda. \tag{1.5}$$

It is easy to see (from (1.4)) that if $f \in C_c^\infty(X)$ with f supported in a ball about the origin of radius R, then $M^a(x)=0$ for $|\log a| > R + d(x, 0)$, d being the Riemannian metric on X, log being the inverse of the exponential map restricted to $\mathfrak{a}$, and $|\cdot|$ being the norm on $\mathfrak{a}$ induced from the Killing form. In Section 4 we obtain a partial converse of this fact, known as a support theorem. A weaker support result for spherical means on real hyperbolic space was proved by Helgason [16].

A brief outline of this paper is as follows. Section 2 contains the necessary preliminaries on Lie groups, representations, and Jacobi functions. Section 3 develops the spectral Paley–Wiener theorem for rank $G=1$ and some variations. Applications of the main result are given in Section 4. Finally, in Section 5 we return to the general rank case for commentary.

2. Preliminaries

This section divides naturally into two parts: (A) general considerations and (B) rank one preliminaries—Jacobi functions.

A. *General Considerations*

In this subsection we provide an overview of structural aspects of semi-simple Lie groups and introduce Helgason's Fourier transform on symmetric spaces. References for this section are [11, 12, 14, 15].

Returning to the notation of Section 1, an element $\gamma \in \mathfrak{a}^*$ is called a root if $\gamma \neq 0$ and $\mathfrak{g}_\gamma = \{X \in \mathfrak{g} \mid [H, X] = \gamma(H)\, X,$ for all $H \in \mathfrak{a}\} \neq \{0\}$ ([,] denotes the Lie bracket). The multiplicity of root γ is het number $m_\gamma = \dim \mathfrak{g}_\gamma$. Let $\mathfrak{a}'$ be the open subset of $\mathfrak{a}$ on which no root vanishes. The components of $\mathfrak{a}'$ are called Weyl chambers. Select a Weyl chamber $\mathfrak{a}_+$; a root γ is called positive if $\gamma(H) > 0$ for all $H \in \mathfrak{a}_+$. We set $\rho = \frac{1}{2} \sum m_\gamma \gamma$, where the sum is over the positive roots. Let M, resp. M', be the centralizer, resp. normalizer, of A in K. The finite group $W = M'/M$, called the Weyl group, acts as a group of linear transformations on $\mathfrak{a}$ which permute the Weyl chambers. (W also acts on $\mathfrak{a}_c^*$ via $w\lambda(H) = \lambda(w^{-1}H)$, $\lambda \in \mathfrak{a}_c^*$, $w \in W$.)

Expressing the Iwasawa decomposition in the form $G = NAK$, any element $g \in G$ has the unique form $g = n(g) \exp(A(g))\, k(g)$, where $A(g) \in \mathfrak{a}$ ($A(g) = -H(g^{-1})$, where H is the function introduced in Section 1). The function $A(k^{-1}g)$ is right-invariant in the g variable under K, and right-invariant in the K variable under M, i.e., defines a function on $K/M \times G/K$. We write $A(x, b) = A(k^{-1}g)$, where $x = gK$ and $b = kM \in K/M$. This function is closely related to the geometry of X (see [8]). Moreover, the function $x \to e^{(i\lambda + \rho)(A(x, b))}$, $\lambda \in \mathfrak{a}_c^*$, is a joint eigenfunction of all invariant differential operators on X. With these ideas in mind, Helgason [11] defined the Fourier transform of an appropriate function f via

$$\tilde{f}(\lambda, b) = \int_X f(x)\, e^{(-i\lambda + \rho)(A(x, b))}\, dx, \tag{2.1}$$

where dx is a Riemannian measure X (the Haar measure dg on G is normalized via $\int_G f(gK)\, dg = \int_X f(x)\, dx$, providing consistency with formulas in Section 1). The inversion formula, valid for $f \in C_c^\infty(X)$, say, is given by

$$f(x) = \frac{1}{(2\pi)^r\, |W|} \int_{\mathfrak{a}_+^*} \int_{\mathbf{B}} \tilde{f}(\lambda, b)\, e^{(i\lambda + \rho)(A(x, b))}\, db\, |c(\lambda)|^{-2}\, d\lambda. \tag{2.2}$$

In this formula $d\lambda$ is the usual Lebesgue measure on $\mathfrak{a}^*$, db is the normalized measure on $\mathbf{B} = K/M$, $r = \dim \mathfrak{a}$ is the rank of G, $|W|$ is the order of the Weyl group, and $c(\lambda)$ is the Harish–Chandra c-function.

The spherical function on X has the form (Harish–Chandra, see [15])

$$\Phi_\lambda(x) = \int_{\mathbf{B}} e^{(i\lambda + \rho)(A(x, b))}\, db \tag{2.3}$$

for $\lambda \in \mathfrak{a}_c^*$ and satisfies

$$\Phi_\lambda(g^{-1}h) = \int_{\mathbf{B}} e^{(-i\lambda+\rho)\,A(gK,\,b)} e^{(-i\lambda+\rho)\,A(hK,\,b)}\,db. \tag{2.4}$$

The latter formula in combination with (2.1) and (1.1) easily gives (1.2). Applying $|A(x, b)| \leqslant d(x, 0)$ to (2.3), we have for $\lambda \in \mathfrak{a}_c^*$

$$|\Phi_\lambda(x)| \leqslant e^{|\mathrm{Im}\,\lambda|\,d(x,\,0)}\,\Phi_0(x). \tag{2.5}$$

On the other hand, for $\lambda \in \mathfrak{a}^*$, one has

$$|\Phi_\lambda(x)| \leqslant \Phi_0(x). \tag{2.6}$$

Write $x = aK$, from the basic estimate [10, p. 161].

$$\Phi_0(a) \leqslant Ce^{-\rho(\log a)}(1 + |\log a|)^s,$$

where C and s are constants; it follows easily that $\Phi_\lambda \in L^q(X)$ for any $q > 2$ provided $\lambda \in \mathfrak{a}^*$. This enable us to define the $P_\lambda f$ for $f \in L^p(X)$ for any $1 \leqslant p < 2$ using (1.1). On the other hand, Helgason's Plancheral theorem for the Fourier transform [11] allows us to define $P_\lambda f$ on $L^2(X)$ via (1.2). These definitions are consistent as (1.1) and (1.2) are equivalent on $C_c^\infty(X)$, a dense subset of $L^p(X)$ for all $p \geqslant 1$. The definitions of $P_\lambda f$ on $L^p(X)$ are of use in Section 4.

B. *Rank One Preliminaries—Jacobi Functions*

Hence forth we assume rank$(G) = 1$. There are two positive roots γ and 2γ (one in the case $G = SO_e(1, n)$ or $SU(1, 1)$). Take $H \in \mathfrak{a}$ with $\gamma(H) = 1$. This allows identification of A with $\mathbf{R}$ and $\alpha_c^*$with $\mathbf{C}$ via the maps $\mathbf{R} \ni t \to \exp(tH) = a_t \in A$ and $\mathbf{C} \ni \lambda \to \lambda\gamma \in \mathfrak{a}_c^*$, respectively. In particular, $\rho = \frac{1}{2}(m_\gamma + 2m_{2\gamma})$.

In the following a key role is played by the irreducible unitary representations of K with M fixed vector, a topic we describe in some detail (see [13]). Let δ be an irreducible unitary representation of K on a vector space V_δ with inner product, $\langle\ ,\ \rangle$ and dimension d_δ. A function Y on K/M is said to be K-finite if the translates of Y under K span a finite-dimensional space denoted $\{K \circ Y\}$. The function Y is said to be K-finite of type δ if the natural representation of K on $\{K \circ Y\}$ decomposes into finitely many copies of δ. Let V_δ^M be the subspace of V_δ fixed under δ/M. Kostant [22] proved that the dimension of V_δ^M is 0 or 1. Let $\hat{K}_0$ be the set of all equivalence classes of irreducible unitary representations δ of K for which $V_\delta^M \neq \{0\}$.

PROPOSITION 2.1 (Helgason [13, 15]). *Let $\delta \in \hat{K}_0$. Let $\{v_1, ..., v_{d_\delta}\}$ be an orthonormal basis of V_δ with $v_1 \in V_\delta^M$. Then the functions $Y_{\delta j}(kM) = \langle v_j, \delta(k)\, v_1\rangle$, $1 \leqslant j \leqslant d_\delta$, form a basis of the space of K-finite functions of type δ on K/M.*

This result enables us to identify V_δ with the space of functions on K/M spanned by $\{Y_{\delta j}\}_{j=1}^{d_\delta}$. The system $\{Y_{\delta j} \mid 1 \leqslant j \leqslant d_\delta, \delta \in \hat{K}_0\}$ satisfies the Schur orthogonally relations and we have the Hilbert space decomposition [12]:

$$L^2(K/M) = \bigoplus_{\delta \in \hat{K}_0} V_\delta.$$

Also note that $Y_{\delta 1}(eK) = 1$ for all $\delta \in \hat{K}_0$ and $Y_{\delta 1}$ is M-invariant.

The following result is somewhat implicit in the work of Helgason [13] and that of Flensted-Jensen and Koornwinder [9]. The result implies group theoretical version of the classic Funk–Hecke theorem for spherical harmonics and is given in the form due to Orloff [19]; variations can be found in [8]. We supply the short proof because of the useful nature of this result in the following.

LEMMA 2.2. *Let dm be the normalized Haar measure on $M(\int_M dm = 1)$. Then for any $Y_\delta \in V_\delta$,*

$$\int_M Y_\delta(kmk_1 M)\, dm = Y_\delta(kM)\, Y_{\delta 1}(k_1 M), \qquad k, k_1 \in K.$$

Proof. For each $k \in K$, the function $k_1 M \to \int_M Y_\delta(kmk_1, M)\, dm$ is of K-finite type δ and M-invariant. Hence this function lies in V_δ^M, i.e.,

$$\int_M Y_\delta(kmk, M)\, dm = g(kM)\, Y_{\delta 1}(k_1 M).$$

Substituting $k_1 = e$ we see that $g(kM) = Y_\delta(kM)$, concluding the proof. ∎

An explicit realization of $\hat{K}_0$ is determined by identifying K/M with the unit sphere in $\mathfrak{p}$. For $m \in \mathbf{Z}^\oplus$, let $\mathscr{H}^m$ be the space of homogeneous harmonic polynomials on $\mathfrak{p}$ of degree m. Identifying the elements of $\mathscr{H}^m$ with their restrictions to the unit sphere we have another Hilbert space decomposition

$$L^2(K/M) = \bigoplus_{m=0}^{\infty} \mathscr{H}^m.$$

Each $\mathscr{H}^m$ decomposes under the natural representation of K into finitely many irreducible subspaces. Effective bookkeeping of the decomposition leads to a parameterization of $\hat{K}_0$ by a pair of integers (p, q). This parametrization was originally given by Kostant [22]; here we use a related parameterization due to Johnson and Wallach [19] and Johnson [18]. In all cases, note that the unit representation of K corresponds to the lattice point $(p, q) = (0, 0)$.

(1) $G = SO_e(1, n)$, $m_\gamma = n-1$, $m_{2\gamma} = 0$, $p \in \mathbf{Z}^+$, $q = 0$;
(2) $G = SU(1, n)$, $m_\gamma = 2n-2$, $m_{2\gamma} = 1$, $(p, q) \in \mathbf{Z}^+ \times \mathbf{Z}$ with $p \pm q \in 2\mathbf{Z}^+$;
(3) $G = \mathrm{Sp}(1, n)$, $m_\gamma = 4m-4$, $m_{2\gamma} = 3$, $(p, q) \in \mathbf{Z}^+ \times \mathbf{Z}^+$ with $p \pm q \in 2\mathbf{Z}^+$;
(4) $G = F_{4(-20)}$, $m_\gamma = 8$, $m_{2\gamma} = 7$, $(p, q) \in \mathbf{Z}^+ \times \mathbf{Z}^+$ with $p \pm q \in 2\mathbf{Z}^+$.

We now turn to the description of the spherical function on X. The algebra of invariant differential operators on X is generated by the Laplace–Beltrami operator L. For $\lambda \in \mathbf{C}$ let $\mathscr{E}_\lambda(X) = \{ f \in C^\infty(X) \mid Lf = -(\lambda^2 + \rho^2) f \}$ and let $\mathscr{E}_{\lambda, \delta}(X)$ be the set of K-finite functions of type δ in $\mathscr{E}_\lambda(X)$. The *spherical function of type* δ is the function $\Phi_{\lambda, \delta} \in \mathscr{E}_{\lambda, \delta}(X)$ given by

$$\Phi_{\lambda, \delta}(X) = \int_K e^{(i\lambda + \rho)(A(x, kM)} Y_{\delta 1}(kM)\, dk.$$

Note that if δ is the unit representation, then $\Phi_{\lambda, \delta}$ is the spherical function Φ_λ on X. More generally, the functions

$$F_{\delta_j}(x) = \int_K e^{(i\lambda + \rho)(A(x, kM)} Y_{\delta, j}(kM)\, dk, \qquad 1 \leqslant j \leqslant d_\delta,$$

form a basis of $\mathscr{E}_{\lambda, \delta}(X)$ proved $\mathrm{Re}(i\lambda) \geqslant 0$ (Helgason [13]). Furthermore, Helgason showed that the space $\mathscr{E}_{\lambda, \delta}(X)$ is determined by $\Phi_{\lambda, \delta}$ in the sense that

$$F_{\delta, j}(ka) = Y_{\delta_{ij}}(kM)\, \Phi_{\lambda, \delta}(a). \tag{2.7}$$

Much of what follows depends on the explicit structural form and integral representations for the function $\Phi_{\lambda, \delta}(t) = \Phi_{\lambda, \delta}(a_t)$. Helgason [13] obtained an explicit expression in terms of hypergeometric functions using a "polar" form for the Laplace–Beltrami operator. This depends on the aforementioned parameterization by a pair of integers (p, q) of the set $\hat{K}_0$.

Set $\alpha = \frac{1}{2}(m_\gamma + m_{2\gamma} - 1)$ and $\beta = \frac{1}{2}(m_{2\gamma} - 1)$. Helgason's formula expressed using Jacobi functions is as follows: for each $\delta \in \hat{K}_0$, there are integers (as above) (p, q) such that:

$$\Phi_{\lambda, \delta}(t) = Q_\delta(i\lambda + \rho)(\alpha + 1)_p^{-1} (\operatorname{sh} t)^p (\operatorname{ch} t)^q\, \phi_\lambda^{(\alpha + p, \beta + q)}(t), \tag{2.8}$$

where $\phi_\lambda^{(\mu,\tau)}$ is the Jacobi function (of the first kind) with parameters (μ, τ), $(z)_m = \Gamma(z+m)/\Gamma(z)$, and Q_δ are Kostant's polynomials,

$$Q_\delta(i\lambda+\rho) = \left(\tfrac{1}{2}(\alpha+\beta+1+i\lambda)\right)_{(p+q)/2} \left(\tfrac{1}{2}(\alpha-\beta+1+i\lambda\right)_{(p-q)/2}. \tag{2.9}$$

(simple computation will verify that the right-hand side of (2.9) is indeed a function of $i\lambda+\rho$). Note that (2.8) for $p=q=0$ becomes the spherical function on X.

Flensted-Jensen and Koornwinder [9] proved an addition theorem for the spherical function, a key result in the methodology of Section 3. Specifically, let $\lambda \in \mathbf{C}$; $t_1, t_2 \in \mathbf{R}$; and $k \in K$, then

$$\Phi_\lambda(\bar{a}_{t_1}^1 k a_{t_2}) = \sum_{\delta \in \hat{K}_0} \Phi_{\lambda,\delta}(a_{t_1})\, \Phi_{\lambda,\delta}(a_{t_2})\, Y_{\delta 1}(kM). \tag{2.10}$$

The series in (2.10) is absolutely and uniformly convergent on compact subsets (in all four variables). The proof of (2.10) also makes implicit use of Lemma 2.2.

Formulas (2.4) and (2.10) have an interesting simplification in the case $G = SO_e(1, n)$. For this set $\Phi_{\lambda,\delta} = \Phi_{\lambda,p}$ and we denote the spherical function on $H^m = SO(1, m)/SO(m)$ by $\Phi_\lambda^{(m)}$. Then (2.4) becomes

$$\Phi_{\lambda,p}(t) = Q_\delta\left(i\lambda + \frac{n-1}{2}\right)\left(\frac{n-2}{2}\right)_p^{-1} (\operatorname{sh} t)^p\, \Phi_\lambda^{(n+2p)}(t), \tag{2.11}$$

a formula relating the spherical function of type "p" to the spherical function on the higher dimensional space H^{n+2p}. This is very reminiscent of ideas underlying the Hecke–Bochner identity [21] for the Fourier transform on Euclidean space, and in fact implies an analog for the Fourier transform on the real hyperbolic space H^n. Such simplification is not available in the other rank-one spaces.

We need to collect a few facts concerning Jacobi functions (Refs. [9, 21]). For our purposes we may assume $\mu, \tau \geqslant -\frac{1}{2}$. The Jacobi function is the unique smooth solution of the differential equation

$$u'' + \frac{\Delta'_{u,\tau}}{\Delta_{\mu,\tau}}\, u' = -(\lambda^2+\rho^2)\, u, \tag{2.12}$$

normalized by $u(0) = 1$. Here $\rho = \mu + \tau + 1$ and $\Delta_{\mu,\tau} = (2 \operatorname{sh} t)^{2\mu+1} \times (2 \operatorname{ch} t)^{2\tau+1}$. If $\mu = \alpha = \frac{1}{2}(m_\gamma + m_{2\gamma} - 1)$ and $\tau = \beta = \frac{1}{2}(m_{2\gamma} - 1)$, the differential operator on the left-hand side of (2.7) is the radial of the Laplace–Beltrami operator on X.

The Jacobi transform of an even function $f \in C_c^\infty(\mathbf{R})$ is given by

$$\tilde{f}(\lambda) = \int_0^\infty f(t)\, \phi_\lambda^{(\mu,\tau)}(t)\, \Delta_{\mu,\tau}(t)\, dt. \tag{2.13}$$

This transform is inverted via

$$f(\lambda) = \frac{1}{2\pi} \int_0^\infty \hat{f}(\lambda)\, \phi_\lambda^{(\mu,\tau)}(t)\, |c_{\mu,\tau}(\lambda)|^{-2}\, d\lambda, \tag{2.14}$$

where

$$c_{\mu,\tau}(\lambda) = \frac{2^{\rho - i\lambda} \Gamma(\mu+1)\, \Gamma(i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+\rho))\, \Gamma\frac{1}{2}(i\lambda+\mu-\tau+1))}. \tag{2.15}$$

Again, where $\mu = \alpha$ and $\tau = \beta$, formulas (2.8), (2.9), and (2.10) are the spherical transform of a radial function on X, the corresponding inversion formula, and the Harish–Chandra c-function for X, respectively. See [21] for an illuminating discussion of the group theoretic interpretations of Jacobi functions.

We have need of the following Paley–Wiener theorem for the Jacobi transform [21].

THEOREM 2.3. *The Jacobi transform is a bijection of the space of even C_c^∞ functions on $\mathbf{R}$ with support $\subset [-R, R]$ $(R > 0)$ onto the space of even entire functions g which satisfy that for each $N = 0, 1, ...,$ there is a constant C_N such that*

$$|g(\lambda)| \leqslant C_N (1 + |\lambda|)^{-N}\, e^{R\,|\mathrm{Im}\,\lambda|}.$$

To conclude this section we use the following technical result.

LEMMA 2.4. *Let $\mu, \tau \geqslant -\frac{1}{2}$, then for any $\Lambda > 2/\pi$, there is a constant $C = C(\Lambda, \mu, \tau)$ such that*

$$|\phi_\lambda^{(\mu,\tau)}(1/|\lambda|^2)| \geqslant C \qquad \textit{for} \quad |\lambda| \geqslant \Lambda.$$

Proof. Again we set $\rho = \mu + \tau + 1$. We need the following integral representation for the Jacobi function (Koornwinder [21])

$$\phi_\lambda^{(\mu,\tau)}(t) = \int_0^1 \int_0^\pi |\mathrm{ch}\, t - re^{i\phi}\, \mathrm{sh}\, t|^{-i\lambda-\rho}\, dM_{\mu,\tau}(t, \phi),$$

where $dM_{\mu,\tau}(r, \phi)$ is a probability measure on $[0, 1] \times [0, \pi]$ (see [16] for the explicit formula). The lemma will be proved by applying elementary estimated to the function $|\text{ch}\, t - re^{-i\phi}\, \text{sh}\, t|^{-i\lambda-\rho}$. The starting point for the analysis is the simple inequality

$$|\phi_\lambda^{(\mu,\tau)}(t)| \geqslant |\text{Re}\, \phi_\lambda^{(\mu,\tau)}(t)|, \tag{2.16}$$

and by simple computation

$$\begin{aligned} &\text{Re}\, |\text{ch}\, t - re^{i\phi}\, \text{sh}\, t|^{-i\lambda-\rho} \\ &\quad = \cos[(\text{Re}\, \lambda)\, \lg |\text{ch}\, t - re^{i\phi}\, \text{sh}\, t|] \times |\text{ch}\, t - re^{i\phi}\, \text{sh}\, t|^{\text{Im}\, \lambda - \rho}. \end{aligned} \tag{2.17}$$

Without loss of generality, we assume $\text{Re}\, \lambda \geqslant 0$. Provided $(\text{Re}\, \lambda)|\lg |\text{ch}\, t - re^{i\phi}\, \text{sh}\, t|| < \pi/2$, the absolute values on the right-hand side of (2.16) can be dropped. Since $\lg |\text{ch}\, t - re^{i\phi}\, \text{sh}\, t| \leqslant t$, the last condition is satisfied if $0 < (\text{Re}\, \lambda)\, t < \pi/2$ (a condition satisfied when $t = 1/|\lambda|^2$ and $|\lambda| \geqslant \Lambda > 2/\pi$). It follows that $\cos[(\text{Re}\, \lambda)\, \lg |\text{ch}\, t - re^{i\phi}\, \text{sh}\, t|] \geqslant \cos(\text{Re}\, \lambda)\, t$ and so at $t = 1/|\lambda|^2$ this term is $\geqslant \cos(\text{Re}\, \lambda/|\lambda|^2) \geqslant \cos(1/|\lambda|) \geqslant \cos(1/\Lambda)$. To handle the other term in (2.17) first note that $|\text{ch}\, t - re^{i\phi}\, \text{sh}\, t|^{-\rho} \geqslant e^{-\rho t}$. Hence at $t = 1/|\lambda|^2$ this term is $\geqslant e^{-\rho/|\lambda|^2} \geqslant e^{-\rho/\Lambda^2}$. If $\text{Im}\, \lambda \geqslant 0$, then

$$\begin{aligned} |\text{ch}\, t - re^{i\phi}\, \text{sh}\, t|^{\text{Im}\, \lambda} &= [\text{ch}^2\, t - 2r \cos\phi\, \text{sh}\, t + r^2\, \text{sh}^2\, t]^{\text{Im}\, \lambda/2} \\ &\geqslant [\text{ch}\, t - r\, \text{sh}\, t]^{\text{Im}\, \lambda} \geqslant e^{-t\, \text{Im}\, \lambda}. \end{aligned}$$

Again at $t = 1/|\lambda|^2$, this term is $\geqslant e^{-\text{Im}\, \lambda/|\lambda|^2} \geqslant e^{-1/|\lambda|} \geqslant e^{-1/\Lambda}$. The case $\text{Im}\, \lambda < 0$ is similar. Putting the estimates together the lemma is proved with $C = e^{-(1+\rho)/\Lambda} \cos(1/\Lambda)$. ∎

3. Paley–Wiener Theorems

Recall the formula for the generalized spectral projection

$$f_\lambda(x) = P_\lambda f(x) = (f * \Phi_\lambda)(x). \tag{3.1}$$

As such, the function $P_\lambda f(x)$ depends on one more additional variable than the original function f. This additional degree of freedom has useful consequences. In particular, sufficient conditions for $f_\lambda(x)$ to be $P_\lambda f(x)$ for some $f \in C_c^\infty(X)$ are somewhat weaker than necessary conditions. We begin with the latter.

Proposition 3.1. *Let $f \in C_c^\infty(X)$ with $\text{supp}\, f \subset \overline{B(0, R)} = \{x \in X \mid d(0, x) \leqslant R\}$. Then $f_\lambda(x) = P_\lambda f(x)$ satisfies the following four conditions:*

(N1) *$f_\lambda(x)$ is smooth on $X \times \mathbf{R}$;*

(N2) *for each $\lambda \in \mathbf{R}$, $\Delta f_\lambda(x) = -(\lambda^2 + \rho^2)\, f_\lambda(x)$;*

(N3) *for each* $x \in X$, *the map* $\lambda \to f_\lambda(x)$ *is an even entire function satisfying the estimate*

$$|f_\lambda(x)| \leqslant C_N(1+|\lambda|^2)^{-N} e^{(R+d(0,x))\,|\mathrm{Im}\,\lambda|}, \tag{3.2}$$

where C_N *is a constant dependent only on* $N=0, 1, 2, ...$;

(N4) *for* $t>0$ *and* $\delta \in \hat{K}_0$ *and any* $Y_\delta \in V_\delta$, *the map* $\lambda \to [Q_\delta(\rho+i\lambda)\, Q_\delta(\rho-i\lambda)]^{-1} \int_K f_\lambda(ka_t)\, Y_\delta(kM)\, dk$ *is entire.*

Proof. Properties (N1) and (N2) are clear. That $\lambda \to P_\lambda f(x)$ is an even entire function follows from (3.1) and the corresponding property of Φ_λ. To obtain (3.2) we note from (2.5) and subsequent discussion that

$$|\Phi_\lambda(g_1^{-1} \circ x)| \leqslant C e^{(R+d(x,0))\,|\mathrm{Im}\,\lambda|}$$

for $d(g_1 \circ 0, 0) \leqslant R$. This estimate combined with the formula $P_\lambda(Lf)(x) = -(\lambda^2+\rho^2)\, P_\lambda f(x)$ easily yields (3.2). The proof of (N4) is based on the polar form of (3.1),

$$P_\lambda f(ka_t) = \int_K \int_0^\infty f(k_1 a_s)\, \Phi_\lambda(a_s^{-1} k_1^{-1} k a_t)\, \Delta(s)\, ds\, dk_1,$$

where $\Delta(s) = (2\,\mathrm{sh}\, s)^{2\alpha+1}\,(2\,\mathrm{ch}\, s)^{2\beta+1}$ (see [15] or [21]). Using M-invariance this can be written

$$P_\lambda f(ka_t) = \int_K \int_0^\infty f(k_1 a_s) \int_M \Phi_\lambda(a_s^{-1} m^{-1} k_1^{-1} k a_t)\, dm\, \Delta(s)\, ds\, dk_1.$$

So, to compute $f_{\lambda,\delta}(t) = \int_K P_\lambda f(ka_t)\, Y_\delta(kM)\, dk$, we apply Lemma 2.2, formula (2.10), and the invariance property $\Phi_{\lambda,\delta}(-t) = \Phi_{-\lambda,\delta}(t)$ to obtain

$$\begin{aligned}
&\int_K \int_M \Phi_\lambda(a_s^{-1} m^{-1} k_1^{-1} k a_t)\, Y_\delta(kM)\, dm\, dk \\
&\quad = \int_K \Phi_\lambda(a_s^{-1} k a_t) \int_M Y_\delta(k_1 m k)\, dm\, dk \\
&\quad = Y_\delta(k_1 M) \int_K \Phi_\lambda(a_s^{-1} k a_t)\, Y_{\delta 1}(kM)\, dk \\
&\quad = \Phi_{\lambda,\delta}(a_s^{-1})\, \Phi_{\lambda,\delta}(a_t)\, Y_\delta(k_1 M) = \Phi_{\lambda,\delta}(-s)\, \Phi_{\lambda,\delta}(t)\, Y_\delta(k_1 M) \\
&\quad = \Phi_{-\lambda,\delta}(s)\, \Phi_{\lambda,\delta}(t)\, Y_\delta(k_1 M).
\end{aligned}$$

Let $f_\delta(s) = \int_K f(k_1 a_s)\, Y_\delta(k_1 M)\, dk_1$. It follows that

$$f_{\lambda,\delta}(t) = \left[\int_0^\infty f_\delta(s)\, \Phi_{-\lambda,\delta}(s)\, \Delta(s)\, ds\right] \Phi_{\lambda,\delta}(t).$$

Using (2.8), we have

$$\begin{aligned}&[Q_\delta(\rho+i\lambda)\, Q_\delta(\rho-i\lambda)]^{-1} f_{\lambda,\delta}(t)\\ &\quad= \left[\int_0^\infty f_\delta(s)\, \phi_\lambda^{(\alpha+p,\beta+q)}(s)(\operatorname{sh} s)^p\, (\operatorname{ch} s)^q\, \Delta(s)\, ds\right]\\ &\qquad\times (\operatorname{sh} t)^p\, (\operatorname{ch} t)^q\, \phi_\lambda^{(\alpha+p,\beta+q)}(t),\end{aligned}$$

where (p, q) are the integers corresponding to δ. Property (N4) follows from this last formula and the fact that $\phi^{(\alpha+p,\beta+q)}(t)$ does not vanish identically in λ for any $t>0$. This concludes the proof. ∎

Our proof of (N4) is based on (3.1) and the addition theorem for the spherical function. This is natural from the perspective of harmonic analysis taken in this paper and should find applicability in other settings, notably non-Riemannian symmetric spaces associated with rank-one semisimple Lie groups. In the Riemannian symmetric spaces X, a slightly simpler (no dependence on (2.10)) proof of (N4) can be obtained using (1.2) and formulas for the Helgason Fourier transform on X.

The "hard part" of the Paley–Wiener theorem for $P_\lambda f$ is the following.

THEOREM 3.2. *Let $f_\lambda(x)$ be continuous on $X\times\mathbf{R}$. Then $f_\lambda(x) = P_\lambda f(x)$ for some $f\in C_c^\infty(X)$ with* supp $f \subset \overline{B(0; R)}$, *provided the following four conditions hold*:

(P1) *for each $\lambda\in\mathbf{R}$, $Lf_\lambda(x) = -(\lambda^2+\rho^2)\, f_\lambda(x)$, weakly*;

(P2) *for each $x\in X$, the map $\lambda\to f_\lambda(x)$ has an even entire extension*;

(P3) *for each $N = 0, 1, 2, \ldots$, there exists a positive increasing continuous function C_N on $(0,\infty)$ such that $|f_\lambda(x)| \leqslant C_N(d(0,x))\,(1+|\lambda|^2)^{-N}\, e^{(R+d(0,x))\,|\mathrm{Im}\,\lambda|}$*;

(P4) *property* (N4) *holds.*

Proof. For $m = 0, 1, \ldots$, let

$$F_m(x) = \frac{(-1)^m}{2\pi} \int_0^\infty (\lambda^2+\rho^2)^m f_\lambda(x)\, |c(\lambda)|^{-2}\, d\lambda \tag{3.3}$$

and set $F_0 = f$. The integrals are absolutely convergent by (P3) ($|c(\lambda)|^{-2}$ has polynomials growth for $\lambda\in\mathbf{R}$) and so $F_m\in C(X)$ for each m. A standard duality computation shows that $L^m f = F_m$ in the weak sense. Hence, by elliptic regularity [17], $f \in C^\infty(X)$. We will show $\operatorname{supp} f \subset \overline{B(0; R)}$ by decomposing f into K-types. Specifically, using (3.3) with $m = 0$, for $\delta \in \hat{K}_0$,

$$f_\delta(t) = \int_K f(ka_t) \, \overline{Y_\delta(kM)} \, dk$$

$$= \frac{1}{2\pi} \int_0^\infty f_{\lambda, \delta}(t) \, |c(\lambda)|^{-2} \, d\lambda, \tag{3.4}$$

$f_{\lambda, \delta}$ being defined analogous to f_δ. It suffices to show $f_\delta(t) = 0$ for $t > R$ and all $\delta \in \hat{K}_0$. The point is that, because of (P1), the function $f_{\lambda, \delta}(t) \, Y_\delta(kM)$ is in $\mathscr{E}_{\lambda, \delta}(X)$. As such, we have the structural form of $f_{\lambda, \delta}$,

$$f_{\lambda, \delta}(t) = h_\delta(\lambda) \, \Phi_{\lambda, \delta}(t), \tag{3.5}$$

for some function $h_\delta = h$. The key technical step is summarized:

(J) The function $g(\lambda) = h(\lambda)/Q_\delta(\rho - i\lambda)$ has even analytic extension satisfying the estimate: for $N = 0, 1, 2, \ldots$, there is a constant C_N such that

$$|g(\lambda)| \leqslant C_N (1 + |\lambda|^2)^{-N} \, e^{R \, |\operatorname{Im} \lambda|}. \tag{3.6}$$

Proof of (J). From (P2), (P3), and the definition of $f_{\lambda, \delta}$ it follows that for each $t \geqslant 0$, the map $\lambda \to f_{\lambda, \delta}(t)$ has even entire extension and satisfies the estimate

$$|f_{\lambda, \delta}(t)| \leqslant C_N(t) (1 + |\lambda|^2)^{-N} \, e^{(R + t) \, |\operatorname{Im} \lambda|}. \tag{3.7}$$

Further, by (P4), the map

$$\lambda \to [Q_\delta(\rho + i\lambda) \, Q_\delta(\rho - i\lambda)]^{-1} f_{\lambda, \delta}(t) \tag{3.8}$$

is entire for $t > 0$. Applying (2.8) in (3.5) we get

$$f_{\lambda, \delta}(t) = [(\alpha + 1)_p^{-1} g(\lambda) [Q_\delta(\rho + i\lambda) \, Q_\delta(\rho - i\lambda)]$$
$$\times (\operatorname{sh} t)^p \, (\operatorname{ch} t)^q \, \phi_\lambda^{(\alpha + p, \beta + q)}(t). \tag{3.9}$$

This formula shows that $g(\lambda)$ is even. In the subsequent argument we suppose δ is not the unit representation; the latter case is considerably simpler. Let Λ be greater than the largest root of the polynomial $Q_\delta(\rho + i\lambda) \, Q_\delta(\rho - i\lambda)$. Clearly $\Lambda > 2/\pi$. Also, note that this polynomial has degree $2p$. By substituting $t = 1/|\lambda|^2$ in (3.9), it follows that $g(\lambda)$ is analytic for $|\lambda| > \Lambda$. By our choice of Λ, there is a constant $d > 0$ such that

$|Q_\delta(\rho+i\lambda)\, Q_\delta(\rho-i\lambda)^{-1}|\,(\mathrm{sh}\,(1/|\lambda|^2))^p \geqslant d$ for $|\lambda| > \Lambda$. Applying estimate (3.7) and Lemma 2.3, we have for some constant d' and $|\lambda| > \Lambda$,

$$\begin{aligned} |g(\lambda)| &\leqslant d' C_N\left(\frac{1}{|\lambda|^2}\right)(1+|\lambda|^2)^{-N}\, e^{(R+1/|\lambda|^2)\,|\mathrm{Im}\,\lambda|} \\ &\leqslant d' e^{1/\Lambda} C_N(1/\Lambda^2)(1+|\lambda|^2)^{-N}\, e^{R\,|\mathrm{Im}\,\lambda|}. \end{aligned} \tag{3.10}$$

Since $\phi_\lambda^{(\alpha+p,\,\beta+q)}(0)=1$ for all $\lambda \in \mathbf{C}$, we can find a sufficiently small $t_0 > 0$ such that $|\phi_\lambda^{(\alpha+p,\,\beta+q)}(t_0)| \geqslant \frac{1}{2}$ for $|\lambda| \leqslant \Lambda+1$. Using (3.8) and (3.9) with $t=t_0$ it follows that $g(\lambda)$ is analytic for $|\lambda| < \Lambda+1$. This together with (3.10) completes the proof of (J).

Applying Theorem 2.3, the function $g(\lambda)$ is the Jacobi transform (with parameters $(\alpha+p, \beta+q)$) of an even $F_\delta \in C_c^\infty(\mathbf{R})$ with $F_\delta(t)=0$ for $t>R$. Using (3.4), (3.5), and (2.8),

$$\begin{aligned} f_\delta(t) &= \frac{1}{2\pi}\int_0^\infty h(\lambda)\, \Phi_{\lambda,\delta}(t)\, |c(\lambda)|^{-2}\, d\lambda \\ &= \frac{1}{2\pi}(\mathrm{sh}\, t)^p\, (\mathrm{ch}\, t)^q \\ &\quad \times \int_0^\infty g(\lambda)\, \phi_\lambda^{(\alpha+p,\,\beta+q)}(t)(\alpha+1)_p^{-1}\, Q_\delta(\rho+i\lambda)\, Q_\delta(\rho-i\lambda)\, |c(\lambda)|^{-2}\, d\lambda. \end{aligned}$$

By (2.15) and (2.9),

$$|c_{(\alpha+p,\,\beta+q)}(\lambda)|^{-2} = Q_\delta(\rho+i\lambda)\, Q_\delta(\rho-i\lambda)\, |c(\lambda)|^{-2}\, (\alpha+1)_p^{-2}.$$

Hence the Jacobi inversion formula (2.14) yields $f_\delta(t) = (\mathrm{sh}\, t)^p\, (\mathrm{ch}\, t)^q\, F_\delta(t)$ and so $f_\delta(t)=0$ for $t>R$, as desired. Finally, to prove $f_\lambda(x) = P_\lambda f(x)$, it will suffice to prove that $\int_0^\infty f_\lambda(x)\, |c(\lambda)|^{-2}\, d\lambda = 0$ for all $x \in X$ implies $f_\lambda(x) \equiv 0$. Again using decomposition of $f_\lambda(x)$ into K-types, this reduces to uniqueness result for the Jacobi transform. This concludes the proof of Theorem 3.2. ∎

Remark. Sufficient conditions for $f_\lambda(x) = P_\lambda f(x)$ for some $f \in C_c^{N_0}(X)$ with $\mathrm{supp}\, f \subset \overline{B(0;R)}$ can be obtained by a slight modification of the first part of the proof. The only change is in (P3) which is replaced by

(P3′) For $N = 0, 1, ..., N_0+q+1$, there exists a positive continuous increasing function C_N on $(0, \infty)$ such that

$$|f_N(x)| \leqslant C_N(d(0,x))(1+|\lambda|^2)^{-N}\, e^{(R+d(0,x))\,|\mathrm{Im}\,\lambda|}.$$

Here $q \in \mathbf{Z}^+$ is the polynomial order of $|c(\lambda)|^{-2}$ on $\mathbf{R}$.

As indicated earlier, Helgason's Paley–Wiener theorem for the Fourier transform (rank one case) is a consequence of Theorem 3.2. Specifically, we have

THEOREM 3.3 (Helgason [12]). *Let* $g(\lambda, b)$ *be continuous on* $\mathbf{R} \times K/M$. *Then* $g(\lambda, b) = \tilde{f}(\lambda, b)$ *for some* $f \in C_c^\infty(X)$ *with* $\operatorname{supp} f \subset \overline{B(0; R)}$ *if and only if*

(H1) *for each* $b \in K/M$, $\lambda \to g(\lambda, b)$ *has analytic extension satisfying the estimates* $|g(\lambda, b)| \leqslant d_N (1 + |\lambda|)^{-N} e^{R\,|\mathrm{Im}\,\lambda|}$ *for each* $N = 0, 1, ...$;

(H2) *the map* $\lambda \to \int_B e^{(i\lambda + \rho)(A(x, b))} g(\lambda, b)\, db$ *is even for each* $x \in X$.

Proof. We prove sufficiency by (H1) and (H2) only. For this define

$$f_\lambda(x) = \int_B g(\lambda, b)\, e^{(i\lambda + \rho)(A(x, b))}\, db. \tag{3.11}$$

It is clear that $f_\lambda(x)$ is continuous on $\mathbf{R} \times \mathbf{X}$ and satisfies (P1) and (P2) in Theorem 3.2. To verify (P3) note that the estimate in (H1) is equivalent with $|g(\lambda, b)| \leqslant d_N (1 + |\lambda|^2)^{-N} e^{R\,|\mathrm{Im}\,\lambda|}$ for all $N = 0, 1, ...$ Applying this and (2.5) in (3.11) we obtain (P3) where $C_N(\,\cdot\,)$ is a constant. To verify (P4) we expand $g(\lambda, kM)$ into K-types, $g(\lambda, kM) \sim \sum g_{\delta, j}(\lambda)\, Y_{\delta, j}(kM)$. Substituting in (3.11) and using group theoretical computations similar to those done earlier, we obtain

$$\begin{aligned} f_{\lambda, \delta, j}(t) &= \int_K f_\lambda(k a_t)\, Y_{\delta j}(km)\, dk \\ &= g_{\delta, j}(\lambda)\, \Phi_{\lambda, \delta}(t). \end{aligned} \tag{3.12}$$

For each t, the function $\lambda \to f_{\lambda, \delta, j}(t)$ is an even entire function. Consequently, using (2.8) we have the invariance property

$$g_{\delta, j}(-\lambda)\, Q_\delta(\rho - i\lambda) = g_{\delta, j}(\lambda)\, Q_\delta(\rho + i\lambda), \qquad \lambda \in C. \tag{3.13}$$

Hence, $q_{\delta, j}(\lambda)/Q_\delta(\rho - i\lambda)$ has removable singularities at the zeros of $Q_\delta(\rho - i\lambda)$, i.e., is entire. Again by (2.8),

$$\begin{aligned} &[Q_\delta(\rho + i\lambda)\, Q_\delta(\rho - i\lambda)]^{-1} f_{\lambda, \delta, j}(t) \\ &\quad = (\alpha + 1)_p^{-1}\, [Q_\delta(\rho - i\lambda)^{-1}\, g_{\delta, j}(\lambda)](\mathrm{sh}\, t)^p\, (\mathrm{ch}\, t)^q\, \phi_\lambda^{(\alpha + p, \beta + q)}(t). \end{aligned}$$

Using arguments similar to those in the Proof of Proposition 3.1, we deduce that the left-hand side is entire for $t > 0$ as desired. Applying Theorem 3.2, $f_\lambda(x) = P_\lambda f(x)$ for some $f \in C_c^\infty(X)$ with $\operatorname{supp} f \subset \overline{B(0; R)}$. That $g = \hat{f}$ is easy, based on (3.12). This concludes the proof. ∎

Remarks. (3.3.1) The invariance property (3.13) is well known [12, 13]. It plays a key role in Helgason's Proof of Theorem 3.3.

(3.3.2) The attentive reader will note that Theorem 3.2 can also be deduced from Theorem 3.3; one must appropriately apply Theorem 3.3 instead of Theorem 2.3. No significant simplification of the proof of Theorem 3.2 occurs, however.

4. Support Theorems

In this section we primarily study the spherical mean operator on $X = G/K$ in the rank one case; variations on the theme are mentioned at the end. Let $f \in C(X)$; the spherical mean of f is defined by [16]:

$$M^h f(g) = \int_K f(gkh)\, dk, \qquad g, h \in G. \tag{4.1}$$

Note that $M^h f(g)$ is K-invariant in the h-variable; thus we often write $M^t f(g)$ in place of $M^h f(s)$, where $h = k_1 a_t k_2$, $t > 0$. For $g \in G$ and $t \in \mathbf{R}^+$, the orbit $\{gka_t K \mid k \in K\}$ is the sphere centered at $x = gK$ of radius t. Geometrically, $M^t f(g) = M^t f(x)$ is the average of f over this sphere with respect to the measure induced by the Riemannian measure on X [16]. It is clear from (4.1) that if $f(x) = 0$ for $d(x, 0) > R$, then $M^t f(x) = 0$ for $t > R + d(0, x)$, for all $x \in X$. A support theorem is a converse of this fact assuming certain regularity/growth on f, i.e., is a Tauberian theorem.

If $f \in C^\infty(X)$, it is clear from (4.1) that $u(x, y) = M^h f(g)$, where $x = gk$, $y = hK$, is C^∞ on $X \times X$. Suppose now that $f \in C_c^\infty(X)$. Using (4.1) and (1.3), and the basic functional relation [15]

$$\Phi_\lambda(g)\, \Phi_\lambda(h) = \int_K \Phi_\lambda(gkh)\, dk, \qquad g, h \in G, \tag{4.2}$$

one obtains a "spectral" form for $M^h f(g)$:

$$M^h f(g) = \frac{1}{2\pi} \int_0^\infty \Phi_\lambda(h)\, P_\lambda f(g)\, |c(\lambda)|^{-2}\, d\lambda. \tag{4.3}$$

In terms of our function u, this is

$$u(x, y) = \frac{1}{2\pi} \int_0^\infty \Phi_\lambda(y)\, P_\lambda f(x)\, |c(\lambda)|^{-2}\, d\lambda. \tag{4.4}$$

Equivalently, if $\hat{u}(x, \lambda)$ denotes the spherical transform of u in the second variable, then

$$\hat{u}(x, \lambda) = P_\lambda f(x). \tag{4.5}$$

If $f \in L^p(X)$ for some $p \geqslant 1$ it is a simple consequence of Fubini's theorem that $M^h f(g)$ is defined for every $g \in G$ and a.e. $h \in G$ and likewise for $M^t f(x)$. Hence the function $u(x, y)$ is defined for every $x \in X$ and a.e. $y \in X$. Further, by the Minkowski inequality for integrals, $u(\cdot, y) \in L^p(X)$ a.e. (y) and $u(x, \cdot) \in L^p(X)$ for every $x \in X$. The following lemma gives formulas which are the key to the main result.

Lemma 4.1. *Let $f \in L^p(X)$ for some $1 \leqslant p < 2$. Then $\hat{u}(x, \lambda) = P_\lambda f(x)$ for all $x \in X$ and $\lambda \in \mathbb{R}^+$ and*

$$P_\lambda[u(x, \cdot)](y) = P_\lambda f(x)\, \Phi_\lambda(y). \tag{4.6}$$

Proof. Since $u(x, \cdot) \in L^p(X)$ for every X, the spherical transform is defined and continuous. Writing $x = g \cdot 0$ and $y = h \cdot 0$, we have by applying the Fubini argument and a change of variables:

$$\begin{aligned}
\hat{u}(x, \lambda) &= \int_G \left[\int_K f(gkh)\, dk \right] \Phi_\lambda(h)\, dh \\
&= \int_G \left[\int_K f(gkh)\, dk \right] \Phi_\lambda(h^{-1})\, dh \\
&= \int_K \int_G f(gkh)\, \Phi_\lambda(h^{-1})\, dh\, dk \\
&= \int_K \int_G f(h)\, \Phi_\lambda(k^{-1}g^{-1}h)\, dh\, dk \\
&= \int_G f(h)\, \Phi_\lambda(g^{-1}h)\, dh = \int_G f(h)\, \Phi_\lambda(h^{-1}g)\, dh \\
&= P_\lambda f(g).
\end{aligned}$$

This establishes (4.5). Formula (4.6) is a consequence of this and the fact that $P_\lambda F(y) = \hat{F}(\lambda)\, \Phi_\lambda(y)$ when F is K-invariant. ∎

In order to apply these formulas in combination with Theorem 3.2 to obtain a support theorem for spherical means, we need a smoothing operation provided by approximate identities on X. In the realm of symmetric spaces, approximate identities seem to have first been introduced by Stanton and Tomas [24]; our notion is a variation on this which seems more natural from a computational point of view.

Let $\psi \in C_c^\infty(\mathbf{R})$ be even, decreasing for $t>0$, with $\operatorname{supp} \psi = [-1, 1]$ and $\int_0^\infty \psi(t)\, \Delta(t)\, dt = 1$. Let $0 < \varepsilon \leqslant 1$, and set

$$\psi_\varepsilon(t) = \varepsilon^{-1} \psi(t/\varepsilon)^{-1}\, \Delta(t/\varepsilon)\, \Delta(t)^{-1}.$$

Then ψ and ψ_ε extend to K-invariant, smooth, compactly supported functions on X, denoted ψ^e and ψ_ε^e, respectively (e.g., $\psi^e(x) = \psi(d(x, 0))$. Note that $\operatorname{supp} \psi_\varepsilon^e = \overline{B(0; 1)}$ and $\int_X \psi_\varepsilon^e = 1$. The following lemma summarizes facts that can be proved using standard technique.

LEMMA 4.2. (i) *If $f \in C_c(X)$, then* $\lim_{\varepsilon \to 0} (f * \psi_\varepsilon^e)(x) = f(x)$, $x \in X$.

(ii) *If $f \in L^p(X)$ for some $1 \leqslant p < \infty$, then $f * \psi_\varepsilon^e \in L^p(X) \cap C^\infty(X)$ and* $\lim_{\varepsilon \to 0} \| f * \psi_\varepsilon^e - f \|_p = 0$.

We need to examine the pointwise behavior of $f * \psi_\varepsilon^e$ when $f \in L^p(X)$. Clerc and Stein [6] introduced a maximal function for locally integrable functions f by

$$Mf(x) = \sup_{0<r<1} |B(x, r)|^{-1} \int_{B(x, r)} |f(y)|\, dy,$$

where $|B(x, r)|$ is the (Riemannian) measure of the ball. In [6] it is shown that the map $f \to Mf$ is weak type (1,1) and strong type (p, p) for $p > 1$. Consequently, if $f \in L^p(X)$ for some $1 \leqslant p < \infty$, the Lebesgue set of f has comeasure zero (a point $x \in X$ is in the Lebesgue set if

$$\lim_{r \to 0} |B(x, r)|^{-1} \int_{B(x, r)} |f(y) - f(x)|\, dy = 0.$$

We have the following variation on a result in [24] for the approximate identity ψ_ε^e.

PROPOSITION 4.3. (i) *If f is locally integrable on X, then* $\sup_{0<\varepsilon<1} |(f * \psi_\varepsilon^e)(x)| \leqslant c Mf(x)$, *where c is a constant.*

(ii) *If $f \in L^p(X)$ for some $1 \leqslant p < \infty$, then* $\lim_{\varepsilon \to 0} (f * \psi_\varepsilon^e)(x) = f(x)$ *for all x in the Lebesgue set of f, in particular almost everywhere.*

Proof. The proof is a modification of that in [24]; for completeness we prove (i) and leave the similar proof of (ii) to the reader. We have

$$|(f*\psi_\varepsilon^e)(x)| = \left| \int_0^\varepsilon M^t f(x)\, \psi_\varepsilon(t)\, \Delta(t)\, dt \right|$$

$$= \left| \int_0^1 M^{\varepsilon t} f(x)\, \Delta(\varepsilon t) \left[\psi(t) \frac{\Delta(t)}{\Delta(\varepsilon t)} \right] dt \right|$$

$$\leqslant \frac{c}{\varepsilon^{2\alpha+1}} \int_0^1 M^{\varepsilon t}\, |f|\, (x)\, \Delta(\varepsilon t)\, \psi(t)\, dt, \tag{4.7}$$

the last step justified by the inequality $\Delta(t)\, \Delta(\varepsilon t)^{-1} \leqslant c\varepsilon^{-(2\alpha+1)}$ for some constant $c > 0$ and $t \in [0, 1]$. Set

$$\Lambda(t) = \int_0^t M^{\varepsilon s}\, |f|\, (x)\, \Delta(\varepsilon s)\, ds = \varepsilon^{-1} \int_0^{\varepsilon t} M^s\, |f|\, (x)\, \Delta(s)\, ds,$$

then applying integration by parts in (4.7) yields

$$|(f*\psi_\varepsilon^e)(x)| \leqslant -\frac{c}{\varepsilon^{2\alpha+1}} \int_0^1 \Lambda(t)\, \psi'(t)\, dt$$

$$\leqslant -\frac{c}{\varepsilon^{2\alpha+2}} \left[\sup_{0<t<1} |B(x; t)|^{-1} \int_{B(x,\, \varepsilon t)} |f(y)|\, dy \right]$$

$$\times \int_0^1 |B(x, t)|\, \psi'(t)\, dt$$

(recall that $\psi' \leqslant 0$). Since $|B(x, t)| = |B(0, t)| = \int_0^t \Delta(s)\, ds$, integration by parts gives

$$\int_0^1 |B(x, t)|\, \psi'(t)\, dt = -\int_0^1 \psi(t)\, \Delta(t)\, dt = -1,$$

and

$$|(f*\psi_\varepsilon^e)(x)| \leqslant \frac{c}{\varepsilon^{2\alpha+2}} \sup_{0<t<1} \frac{1}{|B(x, t)|} \int_{B(x,\, \varepsilon t)} |f(y)|\, dy. \tag{4.8}$$

Since $\varepsilon^{-1} \operatorname{sh} \varepsilon t \leqslant \operatorname{sh} t$ and $\operatorname{ch} \varepsilon t$ is uniformly bounded for $0 < \varepsilon < 1$ and $0 \leqslant t \leqslant 1$,

$$\varepsilon^{-(2\alpha+2)}\, |B(x, \varepsilon t)| = \varepsilon^{-(2\alpha+2)} \int_0^{\varepsilon t} \Delta(s)\, ds = \varepsilon^{-(2\alpha+1)} \int_0^t \Delta(\varepsilon s)\, ds$$

$$\leqslant c' \int_0^t \Delta(c)\, ds = c'\, |B(x, t)|,$$

where c' is a positive constant (recall that $\Delta(t) = (2 \operatorname{sh} t)^{2\alpha+1} (2 \operatorname{ch} t)^{2\beta+1}$). Returning to (4.8) and absorbing constants into c yields

$$|(f * \psi_\varepsilon^e)(x)| \leqslant c \sup_{0<t<\varepsilon} |B(x, t)|^{-1} \int_{B(x, t)} |f(y) \, dy,$$

from which the result follows. ∎

The main result of this section can now be proved.

THEOREM 4.4. *Let $f \in L^p(X)$ for some $1 \leqslant p < 2$. If for some $R > 0$ and all $x \in X$, $M^t f(x) = 0$ for a.e. $t > R + d(x, 0)$, then the map $\lambda \to P_\lambda f(x)$ is entire. Moreover, if for every $s > 0$, $\delta \in \hat{K}_0$, and $Y_\delta \in V_\delta$ the map*

$$\lambda \to [Q_\delta(\rho + i\lambda) \, Q_\delta(\rho - i\lambda)]^{-1} \int_K P_\lambda f(ka_s) \, Y_\delta(k) \, dk \tag{4.9}$$

is continuous on $\mathbf{C}$, then $f(x) = 0$ for a.e. x with $d(x, 0) > R$.

Proof. Let ψ_ε^e be an approximate identity as defined above. Then $f_\varepsilon = f * \psi_\varepsilon \in L^p(X) \cap C^\infty(X)$, the function $u_\varepsilon(x, y) = M^h f_\varepsilon(g)$ $(x = g \cdot 0,\ y = h \cdot 0)$ is smooth in each variable, and it is easily shown that for every $x \in X$, $u(x, y) = 0$ for all y with $d(y, 0) > R + \varepsilon + d(x, 0)$. By Proposition 4.3, it suffices to show that $f_\varepsilon(x) = 0$ for all x with $d(x, 0) > R$. The spherical transform of u_ε in the y variable, $\hat{u}_\varepsilon(x, \lambda)$, is entire for each $x \in X$ and satisfies the estimates

$$|\hat{u}_\varepsilon(x, \lambda)| \leqslant d_N e^{Rd(x, 0)} (1 + |\lambda|^2)^{-N} \, e^{(R + d(x, 0)) \, |\operatorname{Im} \lambda|},$$

where d_N is a constant dependent on $N = 0, 1, 2, \ldots$. By Lemma 4.1, $\hat{u}_\varepsilon(x, \lambda) = P_\lambda f_\varepsilon(x)$, so that $\lambda \to P_\lambda f_\varepsilon(x)$ is entire for each $x \in X$ and satisfies the above estimate. Obviously, $\lim_{\varepsilon \to 0} \hat{u}_\varepsilon(x, \lambda) = \hat{u}(x, \lambda)$.

Further, $\lim_{\varepsilon \to 0} P_\lambda f_\varepsilon(x) = P_\lambda f(x)$, applying Proposition 4.2(ii) to the inequality

$$\begin{aligned}
|P_\lambda f_\varepsilon(x) - P_\lambda f(x)| &= \left| \int_G [f_\varepsilon(g_1) - f(g_1)] \, \Phi_\lambda(g_1^{-1} g) \, dg_1 \right| \\
&\leqslant \|f_\varepsilon - f\|_p \, \|\Phi_\lambda\|_q \\
&\leqslant \|\Phi_0\|_q \, \|f_\varepsilon - f\|_p
\end{aligned}$$

(since $|\Phi_\lambda(g)| \leqslant |\Phi_0(g)|$). Hence the map $\lambda \to P_\lambda f(x)$ has the entire extension as desired. Since $P_\lambda f_\varepsilon(x) = \hat{\psi}_\varepsilon(\lambda) \, P_\lambda f(x)$ we are in a position to apply Theorem 3.2 (f replaced by f_ε), as our hypothesis (4.9) implies (P4) of that theorem. This concludes the proof. ∎

Remarks. (1) The reader should note that the entire argument depends on the fact that the spherical function $\Phi_\lambda \in L^q(X)$ for every $q > 2$. It is unknown if Theorem 4.4 is valid for $f \in L^2(X)$. Given a suitable L^2-theory for the generalized spectral projections $P_\lambda f$, one could hope to develop an analog of Theorem 4.4 for $L^2(X)$. We refer the reader to [26] and [3] for attempts in this direction.

(2) Formula (4.6) of Lemma 4.1 shows that no conclusion about the validity of (P4) in Theorem 3.2 can be deduced from the hypothesis that $u(x, y) = 0$ for a.e. y with $d(y, 0) > R + d(x, 0)$. Consequently, it is unlikely that (4.9) can be eliminated or even weakened in Theorem 4.4.

In order to gain insight into the nature of the hypothesis (4.9) we have the following corollaries. Note that the condition

$$\int_X |f(x)|\, e^{\rho\, d(x, 0)}\, dx < \infty \tag{4.10}$$

implies that $f \in L^1(X, dx)$, the Fourier transform $\tilde{f}(\lambda, b)$ exists as an absolutely convergent integral, and $\tilde{f}(\lambda, b)$ is a continuous function.

COROLLARY 4.5. *Let $f \in C(X)$ satisfy* (4.10). *Suppose that for every $x \in X$, $M^t f(x) = 0$ for $t > R + d(x, 0)$. If for every $b \in B$ the map*

$$\lambda \to \tilde{f}(\lambda, b) \tag{4.11}$$

has an entire extension, then $f(x) = 0$ for $d(x, 0) > R$.

Proof. Without loss of generality, we may suppose $f \in C^\infty(X)$ (convolve with an approximate identity as before). It suffices to verify (4.9) or equivalently (P4) of Theorem 3.2. Expand $\tilde{f}$ into K-types,

$$\tilde{f}(\lambda, k_1 M) = \sum_{\delta, j} \tilde{f}_{\delta, j}(\lambda)\; Y_{\delta, j}(k_1 M),$$

the series being uniformly convergent on K/M for each λ. Then, using (1.2), Lemma 2.2, and (2.7),

$$\begin{aligned} P_\lambda f(ka_t) &= \sum_{\delta, j} \tilde{f}_{\delta, j}(\lambda) \int_K Y_{\delta, j}(kM)\, e^{(i\lambda + \rho)(A(ka_t K, k_1 M))}\, dk_1 \\ &= \sum_{\delta, j} \tilde{f}_{\delta, j}(\lambda)\; \Phi_{\lambda, \delta}(t)\; Y_{\delta, j}(kM). \end{aligned}$$

Hence,

$$I_{\lambda,\delta}(t)=\int_K P_\lambda f(ka_t)\, Y_\delta(kM)\, dk=\tilde{f}_\delta(\lambda)\, \Phi_{\lambda,\delta}(t),$$

where we have dispensed with the nuance indice j. Substituting (2.8), we have

$$I_{\lambda,\delta}(t)=(\alpha+1)_p^{-1}\tilde{f}_\delta(\lambda)\, Q_\delta(\rho+i\lambda)(\operatorname{sh} t)^p\,(\operatorname{ch} t)^q\, \phi_\lambda^{(\alpha+p,\,\beta+q)}(t),$$

where (p, q) are the integers associated with δ. Once again, as in the proof of Theorem 3.3, the fact that $\lambda\to I_{\lambda,\delta}(t)$ is an even entire function implies the invariance property in $\mathbf{C}$:

$$\tilde{f}_\delta(\lambda)\, Q_\delta(\rho+i\lambda)=\tilde{f}_\delta(-\lambda)\, Q_\delta(\rho-i\lambda).$$

Then (4.9) follows also since $\phi_\lambda^{(\alpha+p,\,\beta+q)}(t)$ does not vanish identically in λ for any $t>0$, concluding the proof. ∎

COROLLARY 4.6. *Let $f\in C(X)$ such that for all $x\in X$, $M^tf(x)=0$ for $t>R+d(x,0)$. If for every $l=0, 1, 2, ...,$*

$$\sup_{x\in X} e^{l\,d(x,0)}\,|f(x)|<\infty, \tag{4.12}$$

then $f(x)=0$ for $d(x,0)>R$.

Proof. The hypothesis (4.12) implies (4.10) and (4.11) using the fact that $|A(x,kM)|\leqslant d(x,0)$. ∎

Remark. Corollary 4.6 was proved by Helgason [15] using other techniques for the case X being real hyperbolic space. Helgason used his support theorem for spherical means to obtain a support result for the totally geodesic Radon transform on a real hyperbolic space.

Formula (4.3) suggests the following generalization. Let $\mu, \tau\geqslant -1/2$ and let $f\in C_c^\infty(X)$, and consider the operator

$$M^t_{\mu,\tau}f(x)=\frac{1}{2\pi}\int_0^\infty \phi_\lambda^{(\mu,\tau)}(t)\, P_\lambda f(x)\, |c(\lambda)|^{-2}\, d\lambda. \tag{4.13}$$

When $\mu=\alpha$ and $\tau=\beta$ this becomes (4.3). A "parity" assumption is needed on the parameters (μ, τ) in order to obtain a support result. Specifically one needs $p(\lambda)=|c_{\mu,\tau}(\lambda)|^2\,|c(\lambda)|^{-2}$ to be a polynomial. To conclude this section we give an example which contains the necessary ideas to handle the general case of (4.13).

Let $H^{2m+1}=SO_e(1, 2m+1)/SO(2m+1)$, an odd-dimensional real hyperbolic space, and consider the Cauchy problem for the wave equation:

$$\begin{aligned} u_{tt} &= (\Delta-\rho^2)\,u, \qquad u=u(x,t),\ x\in H^{2n+1},\ t>0. \\ u(x,0) &= 0 \\ u_t(x,0) &= f(x). \end{aligned} \tag{W}$$

If $f\in C_c^\infty(H^{2m+1})$, the solution of (W) may be obtained using separation of variables,

$$u(x,t)=\frac{1}{2\pi}\int_0^\infty \frac{\sin\lambda t}{\lambda}\,P_\lambda f(x)\,|c_m(\lambda)|^{-2}\,d\lambda,$$

where $|c_m(\lambda)|^{-2}=2^{2-4m}\pi\Gamma(m)^2\prod_{l=0}^{m-1}(\lambda^2+l^2)$. The Jacobi function with parameters (1/2, 1/2) is $\phi_\lambda^{(1/2,\,1/2)}(t)=(2\sin 2t)/(\lambda\,\mathrm{sh}\,2t)$, hence

$$U(x,t)=\frac{u(x,t)}{\mathrm{sh}\,t\,\mathrm{ch}\,t}=\frac{1}{2\pi}\int_0^\infty \phi_\lambda^{(1/2,\,1/2)}(t)\,P_\lambda f(x)\,|c_m(\lambda)|^{-2}\,d\lambda, \tag{4.14}$$

an expression of form (4.13). Also, using (2.15),

$$P(\lambda)=|c_{1/2,\,1/2}(\lambda)|^2\,|c_m(\lambda)|^{-2}=2^{4-4m}\pi\Gamma(m)^2\prod_{l=1}^{m-1}(\lambda^2+l^2),$$

a polynomial. If $\operatorname{supp} f\subset\overline{B(0;R)}$, then by Proposition 3.1 and Theorem 2.3, $U(x,t)=0$ for $t>R+d(x,0)$. This latter fact is a version of Huggen's principal for H^{2m+1}. Helgason [16] provided a converse under the hypothesis that $f\in C(H^{2m+1})$ and satisfies (4.12). Here we give a generalization based on the methods of Theorem 4.6.

Two key observations must be made in order to carry over the ideas of Theorem 4.4. We need to extend the definition of U to functions $f\in L^p(X)$ $(1\leqslant p<2)$ and we need an analog of (4.5). The extensions to $L^p(H^{2m+1})$ is relatively easy. The Jacobi function $\phi_\lambda^{(1/2,\,1/2)}(t)$ is an even entire function of λ for each $t>0$ and satisfies the estimate $|\phi_\lambda^{(1/2,\,1/2)}(t)|\leqslant Ce^{t\,|\mathrm{Im}\,\lambda|}$. By a distributional version of Theorem 3.3 (see [12]), for each t, $\phi_\lambda^{(1/2,\,1/2)}(t)$ is the spherical transform of a distribution h_t on H^{2m+1} with support in $\overline{B(0,t)}$. By (4.14) we have

$$U(x,t)=(f*h_t)(x) \tag{4.15}$$

$(f\in C_c^\infty(H^{2m+1}))$. The extension of U or $M^t_{1/2,\,1/2}$ to $L^p(H^{2m+1})$ is now clear by viewing $f\in L^p(H^{2m+1})$ as a distribution.

We can understand the distribution $h_t(\cdot)$ by applying the above ideas to the spherical function

$$\Phi_\lambda(a_t)=\Phi_\lambda(t)=\phi_\lambda^{(\alpha,\beta)}(t),$$

where $\alpha=m-1/2$, $\beta=-1/2$ for H^{2m+1}. Formula (2.3) in this setting gives the interpretation of $\Phi_\lambda(\cdot)$ as the spherical transform of unnormalized surface measure $d\mu_t$ (induced from the Riemannian measure on H^{2m+1}) on the sphere $\{x\in H^{2m+1}\mid d(x,0)=t\}$. Applying Koornwinder's [21] formula for the Jacobi function $\phi_\lambda^{(\alpha,\beta)}(t)$, i.e.,

$$\Phi_\lambda(t)=\frac{2^m\Gamma(m+1/2)}{\pi^{1/2}\Gamma(m)(\operatorname{sh} t)^{2m-1}}\int_0^t \cos\lambda s[\operatorname{ch} t-\operatorname{sh} s]^{m-1}\,ds,$$

we get

$$\begin{aligned}\phi_\lambda^{(1/2,1/2)}(t)=\pi^{1/2}2^{-m}\Gamma(m+1/2)^{-1}(\operatorname{sh} t\operatorname{ch} t)^{-1}&\\ \times\left(\frac{d}{d\operatorname{ch} t}\right)^{m-1}[(\operatorname{sh} t)^{2m-1}\,\Phi_\lambda(t)].&\end{aligned}$$

Consequently, from (4.3), (4.4), and (4.14) we have for $f\in C_c^\infty(H^{2m+1})$,

$$\begin{aligned}U(x,t)=M^t_{(1/2,1/2}f(x)=\pi^{1/2}2^{-m}\Gamma(m+1/2)^{-1}&\\ \times(\operatorname{sh} t\operatorname{ch} t)^{-1}\left(\frac{d}{d(\operatorname{ch} t)}\right)^{m-1}[(\operatorname{sh} t)^{2m-1}\,M^tf(x)].&\end{aligned}$$

This formula was obtained in [16] by other methods and gives the interpretation of $h_t(\cdot)$ as a differential operator (in t) applied to the measure $d\mu_t$. Note that $h_t(\cdot)$ is supported on the sphere $\{x\mid d(0,x)=t\}$, the geometric reason behind our parity assumption on $p(\lambda)$.

Denote by $\hat{u}^{1/2}(x,\lambda)$ the Jacobi transform with parameters (1/2, 1/2) of U in the t-variable. From (4.14), an analog of (4.5) immediately valid for $f\in C_c^\infty(H^{2m+1})$ is

$$\hat{u}^{1/2}(x,\lambda)=p(\lambda)\,P_\lambda f(x). \tag{4.16}$$

The extension of this formula to $L^p(H^{2m+1})$ requires an approximate identity argument.

Lemma 4.7. *Let $f\in L^p(H^{2m+1})$ for some $1\leqslant p<2$. Suppose for some $R>0$ and every $x\in H^{2m+1}$,* supp $U(x,\cdot)\subset[0,R+d(\lambda,0]$. *Then* (4.16) *holds for all $x\in H^{2m+1}$ and $\lambda\in\mathbb{R}^+$.*

Proof. Let ψ_ε^e be an approximate identity on H^{2m+1} and set $f_\varepsilon = f * \psi_\varepsilon^e \in C^\infty(H^{2m+1})$. Then $U_\varepsilon(x,t) = (f_\varepsilon * h_t)(x)$ is C^∞ in each variable and converges to $U(x,t)$ distributionally for each $x \in H^{2m+1}$, $\operatorname{supp} U_\varepsilon(x,\cdot) \subset [0, R+\varepsilon+d(x,0)]$. It follows that

$$\hat{U}_\varepsilon^{1/2}(x,\lambda) = p(\lambda)\, P_\lambda f_\varepsilon(x).$$

Since $P_\lambda f_\varepsilon(x) = \hat{\psi}_\varepsilon^e(\lambda)\, P_\lambda f(x)$, we have $\lim_{\varepsilon\to 0} P_\lambda f_\varepsilon(x) = P_\lambda f(x)$ ($\hat{\psi}_\varepsilon^e(\lambda) = \int_0^\infty \psi(t)\, \Phi_\lambda(\varepsilon t)\, \Delta(t)\, dt$) and the result follows. ∎

We now have the main ingredients for the following result; the proof is left to the reader.

THEOREM 4.8. *Let $u(x,t)$ be the distributional solution of (W) where $f \in L^p(H^{2m+1})$ for some $1 \leqslant p < 2$. Suppose for some $R > 0$ and all $x \in H^{2m+1}$ that* $\operatorname{supp} u(x,\cdot) \subset [0, R+d(x,0)]$. *Then the map $\lambda \to P_\lambda f(x)$ has entire extension to $C - \{\pm il \mid l = 1, 2, ..., m-1\}$. Moreover, if for every $s > 0$, $\delta \in \hat{K}_0$ and $Y_\delta \in V_\delta$, the map*

$$\lambda \to [Q_\delta(\rho+i\lambda)\, Q_\delta(\rho-i\lambda)]^{-1} \int_K P_\lambda f(ka_s)\, Y_\delta(k)\, dk$$

is continuous on $\mathbf{C}$ *then* $\operatorname{supp} f \subset \overline{B(0;R)}$.

5. COMMENTARY

Here we consider $X = G/K$ where G is a connected semisimple Lie group with finite center and any rank. Using the methodology of Section 3 we have

PROPOSITION 5.1. *Let $f_\lambda(x)$ be a continuous function on $\mathfrak{a}^* \times X$, K-invariant in the x-variable. Then $f_\lambda(x) = P_\lambda f(x)$ for some K-invariant $f \in C_c^\infty(X)$ with* $\operatorname{supp} f \subset \overline{B(0;R)}$ *if and only if the following three conditions hold*:

(P′1) *for each $\lambda \in \mathfrak{a}^*$, $x \to f_\lambda(x)$ is an eigenfunction of every invariant differential operator on X, eigenvalues the same as $\Phi_\lambda(x)$*;

(P′2) *for each $x \in X$, the map $\lambda \to f_\lambda(x)$ has Weyl invariant holomorphic extension to $\mathfrak{a}_c^*$*;

(P′3) *for each $N = 0, 1, 2, ...,$ there is a constant C_N such that $|f_\lambda(x)| \leqslant C_N(1+|\lambda|^2)^{-N} e^{(R+d(x,0))\,|\mathrm{Im}\,\lambda|}$.*

The point is that $f_\lambda(x)$ being K-invariant and (P′1) imply that $f_\lambda(x) \in \mathscr{E}_\lambda(X)$. Once again we know the structural form: $f_\lambda(x) = g(\lambda)\, \Phi_\lambda(x)$, for a suitable function g on $\mathfrak{a}^*$. Since $\Phi_\lambda(0) = 1$, we easily deduce properties of g from (P′2) and (P′3), and the Paley–Wiener theorem for the spherical transform on X [15] finishes the proof (in fact, Proposition 5.1 is equivalent to this Paley–Wiener theorem). One can also easily weaken (P′3) in the vein of (P3) of Theorem 3.2 on the sufficiency side of the proposition.

The difficulty in extending Proposition 5.1 to the non-K-invariant case lies in the lack of local analytic/structural information about the spherical functions of type δ. But this we mean analytic/structural information about $\Phi_\lambda(a_t)$ for t near the walls of the Weyl chambers in $\mathfrak{a}$. In the rank-one case such information is immediate from the explicit formulas (2.8).

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